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Quantumness, Generalized 2-Design and Symmetric Informationally Complete POVM

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C. A. Fuchs and M. Sasaki defined the quantumness of a set of quantum states in [1], which is related to the fidelity loss in transmission of the quantum states through a classical channel. In [4], Fuchs showed that in d-dimensional Hilbert space, minimum quantumness is $\frac{2}{d+1}$, and this can be achieved by all rays in the space. He left an open problem, asking whether fewer than d^2 states can achieve this bound. Recently, in a different context, A. J. Scott introduced a concept of generalized t-design in [2], which is a natural generalization of spherical t-design. In this paper, we show that the lower bound on the quantumness can be achieved if and only if the states form a generalized 2-design. As a corollary, we show that this bound can be only achieved if the number of states are larger or equal to d^2, answering the open problem. Furthermore, we also show that the minimal set of such ensemble is Symmetric Informationally Complete POVM(SIC-POVM). This leads to an equivalence relation between SIC-POVM and minimal set of ensemble achieving minimal quantumness.

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1 Introduction

One of the major discrepancy between classical physics and quantum mechanics is the difference in the measurement procedure. In classical physics, in principle, we can perform arbitrarily precise measurement. However, in the realm of quantum mechanics, this is prohibited by the laws of nature. For instance, if we want to measure the momentum and the position of a particle, we cannot measure them simultaneously with an indefinite accuracy due to the following relation.

$$\Delta x \Delta p \geq \frac{\hbar}{2}$$

Then, is there any good measure of 'how quantum' a system is? There is a wide belief - although it has not been explicitly proven - that the notion of quantumness can be only defined on an ensemble of states.[1] For instance, assume we have a single normalized state $|\psi_1\rangle$, and we want to measure 'how quantum' this state is. If we choose an appropriate set of measurement operators(i.e. the POVM trivially constructed from the projectors of the orthonormal set of basis containing $|\psi_1\rangle$), we can measure the state with certainty. That

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is, there is no quantum mechanical effect that differs from the classical mechanics if we use the optimized measurement. However, if we have an ensemble of states which has nonzero overlap between them, the probability of making the wrong guess becomes strictly larger than 0 even if we use the optimal strategy. Based on this idea, the notion of quantumness was first proposed by C. A. Fuchs and M. Sasaki [1] in 2003 with several open problems. In this paper, We will use the definition of the quantumness as the one appearing in [1]. Readers should be always aware that the quantumness is an inverted measure. That is, higher quantumness corresponds to higher classicality.

After his work with M. Sasaki in 2003, C. A. Fuchs showed that symmetric informationally complete ensemble produces a minimal quantumness[4] and left an open problem which asked, "Is it possible to construct an ensemble that generates a minimal quantumness with strictly less than d^2 elements?" In this paper, We will give the answer to the question. It turns out that we need at least d^2 ensemble of states to achieve the minimal quantumness, where d is the dimension of the Hilbert space. Furthermore, the corresponding ensembles form SIC-POVM.

The remainder of the paper will have the following structure. In Section 2, we will introduce several definitions that leads to a definition of quantumness, with some preliminary results. This section will be mostly based on [1]. In Section 3, we will have a brief introducion to frame theory and several new definitions such as *t-extended density operator* and *generalized t-design*. This section will be mainly based on [2] and [3], with some generalizations. Section 4 will be the core of this paper. Here we will prove that ensembles which are generalized 2-design achieves minimal quantumness and vice versa. Also, as a corollary, we will show that minimal number of states to achieve minimal quantumness is d^2. In section 5, we will show that generalized 2-design with minimal elements are SIC-ensemble, thereby showing the connection between SIC-POVM and minimal ensembles achieving the minimal quantumness. Finally, conclusion with remaining open problem will be discussed in Section 6. Throughout the paper, the Hilbert space is assumed to be d dimension unless it is specified otherwise.

2 Definitions and Preliminary Results

Alice sends a quantum message $\mathcal{P} = \{\Pi_i, \pi_i\}$ to Bob. $\Pi_i = |\psi_i\rangle\langle\psi_i|$ where each $|\psi_i\rangle$ is normalized. π_i is the probability of appearance for each quantum states. In between Alice and Bob, the eavesdropper Eve intercepts the message. To do so, Eve has to measure the states that is sent from Alice and must reconstruct it. Eve's measurement is denoted as $\mathcal{E} = \{E_b\}$, where E_b forms a POVM. Her state reconstruction strategy is $\mathcal{M}$, which is a mapping $b \rightarrow \sigma_b$. Here σ_b is a density matrix. The *average fidelity* $F(\mathcal{E}, \mathcal{M})$ is the fidelity averaged over all possibilities.

$$F_{\mathcal{P}}(\mathcal{E}, \mathcal{M}) = \sum_{b,i} p(b,i)\mathrm{Tr}(\Pi_i\sigma_b) \tag{1}$$

$$= \sum_{b,i} \pi_i \mathrm{Tr}(\Pi_i E_b)\mathrm{Tr}(\Pi_i\sigma_b) \tag{2}$$

The maximal fidelity that can be achieved by changing Eve's strategy is defined as the *accessible fidelity.*

$$F_{\mathcal{P}} = \sup_{\mathcal{E}} \sup_{\mathcal{M}} F_{\mathcal{P}}(\mathcal{E}, \mathcal{M}) \tag{3}$$

By changing the probabilities π_i, one can minimize the *accessible fidelity.* This is the quan-

tumness of the state ensemble $\{\Pi_i\}$.

$$Q_{\{\Pi_i\}} = \inf_{\mathcal{P}} F_{\mathcal{P}} \tag{4}$$

Since the fidelity can be made arbitrarily close to 1 for orthogonal states, the maximum quantumness is 1, which corresponds to the classical case. C. A. Fuchs and M. Sasaki proved that the smallest quantumness that can be achieved in d-dimensional Hilbert space is $\frac{2}{d+1}$, and one example of such ensemble is the unitarily invariant ensemble.[1] In his later work, he showed that there exists a discrete ensemble that achieves the smallest quantumness with d^2 elements, which is a symmetric informationlly complete ensemble(SIC-ensemble).[4]

SIC-ensemble is a uniformly distributed ensemble of d^2 normalized vectors in a d-dimensional Hilbert space $\mathcal{H}_d$, satisfying the following condition.

$$\begin{aligned} \mathrm{Tr}(\Pi_i\Pi_j) &= \tfrac{1}{d+1} \quad \text{for} \quad i \neq j \\ &= 1 \quad\quad \text{for} \quad i = j \end{aligned} \tag{5}$$

The SIC-ensemble was coined from a symmetric informationally complete POVM (SIC-POVM), and there are many papers about this set of states.[3][5][6] One remarkable fact is that the subnormalized projection operators of these states($\frac{1}{d}\Pi_k$) form a POVM.[5] However, note that the existence of such states in all dimensions has not been proven yet, even though it is highly believed to be so.[3]

3 Frame Theory and Generalized t-design

Frame is a generalization of basis sets. For a d-dimensional Hilbert space, a collection of vectors $|\psi_k\rangle$ is a frame if there exist constants $0 < a \leq b < \infty$ such that

$$a\,|\langle\phi|\phi\rangle|^2 \leq \sum_k |\langle\phi|\psi_k\rangle|^2 \leq b\,|\langle\phi|\phi\rangle|^2 \tag{6}$$

for all $|\phi\rangle$ in the Hilbert space. a and b are called as *frame bounds*, and the frame is *tight* if $a = b$. Now we define the frame operator S.

$$S = \sum_k |\psi_k\rangle\langle\psi_k| \tag{7}$$

where $|\psi_k\rangle$ is a state that lives in the Hilbert space.

Generalized t-design is a natural generalization of spherical t-design, in a sense that it allows the set of states to be uncountable and the probability distribution to be nonuniform. Note that spherical t-design is a set $\mathcal{D}_t = \{|\psi_k\rangle\}$ of normalized vectors that satisfies

$$\mathrm{Tr}(\otimes_{j=1}^t A_j \frac{1}{|\mathcal{D}_t|}(|\psi_k\rangle\langle\psi_k|)^{\otimes t}) = \mathrm{Tr}(\otimes_{j=1}^t A_j \int_{\mathcal{S}_d} d\psi(|\psi\rangle\langle\psi|)^{\otimes t}) \quad \forall A_j, \tag{8}$$

where $\mathcal{S}_d$ is a set of normalized states in the d-dimensional Hilbert space.

Similarly, *generalized t-design* can be defined as a set of states $\mathcal{D}_t$ together with probability distribution P over $\mathcal{D}_t$ such that $\int_{\mathcal{D}_t} dP = 1$ that satisfies the following.

$$\mathrm{Tr}(\otimes_{j=1}^t A_j \int_{\mathcal{D}_t} dP(\psi)(|\psi\rangle\langle\psi|)^{\otimes t}) = \mathrm{Tr}(\otimes_{j=1}^t A_j \int_{\mathcal{S}_d} d\psi(|\psi\rangle\langle\psi|)^{\otimes t}) \quad \forall A_j, \tag{9}$$

Therefore, $(\mathcal{D}_t, P)$ is a generalized t-design if and only if $\int_{D_t} dP(\psi) \left|\psi\right\rangle \left\langle\psi\right| = \int_{\mathcal{S}_d} d\psi \left|\psi\right\rangle \left\langle\psi\right|$. In [2], the author uses a different definition of generalized t-design, which is shown to be equivalent with the definition of this paper. He shows this by proving

$$\int_{\mathcal{S}_d} d\psi \left|\psi\right\rangle \left\langle\psi\right| = \frac{t!(d-1)!}{(t+d-1)!}\Pi_{sym}, \tag{10}$$

where Π_{sym} is a projector onto a symmetric subspace in $\mathcal{H}_d^{\otimes t}$. Note that the lefthand side of Eq.10 is invariant under the conjugation of $U^{\otimes t}$, where $U \in U(d)$. By Schur's lemma, this must be a multiple of identity transformation on a symmetric subspace. Imposing a trace condition on both sides, we arrive at Eq.10.

Let us define a frame operator with trace 1 as a *unit-trace frame operator*. From any S, we can trivially construct $S' = \frac{S}{\mathrm{Tr}(S)}$ which is a *unit-trace frame operator*. Since this operator is positive, one can see that there exists an equivalent density operator. From this, we can generalize the theorem from Benedetto and Fickus[7].

Theorem 1 *Let S be a unit-trace frame operator constructed from $\{\left|\psi_k\right\rangle\}_{k=1}^n$, where each $\left|\psi_k\right\rangle \in \mathcal{H}_d$. Then the following inequality holds.*

$$\mathrm{Tr}(S^2) \geq \max(\frac{1}{n}, \frac{1}{d}) \tag{11}$$

The bound is achieved if and only if $\{\left|\psi_k\right\rangle\}$ consists of orthogonal vectors with uniform norm, when $n \leq d$, or is a tight frame, when $n \geq d$.

Proof Let us denote the eigenvalues of S as λ_k in a nonincreasing order. The number of eigenvalues are at most $l = \min(n, d)$. Notice the following equation.

$$\mathrm{Tr}(S^2) = \sum_{k=1}^{n} \lambda_k^2 \tag{12}$$

Under the constraint $\sum_{k=1}^n \lambda_k = 1$ and $\lambda_k \geq 0$ for all k, the equality holds if and only if $\lambda_k = \frac{1}{l}$ for all k. For $n \leq d$, $S = \frac{\Pi_n}{n}$, where Π_n is a projector onto a n-dimensional subspace. Therefore, the vectors must be orthogonal to each other and have uniform norm. For $n \geq d$, $S = \frac{1}{d}I$, implying $\{\left|\psi_k\right\rangle\}$ is a tight frame. One can also see that $dS = I$, which means $\{d\left|\psi_k\right\rangle \left\langle\psi_k\right|\}_{k=1}^n$ forms a POVM.

Now we define a *t-extended density operator*. Assume we have a *unit-trace frame operator* $S = \sum_k \pi_k \left|\psi_k\right\rangle \left\langle\psi_k\right|$ from ensemble $\{\Pi_i, \pi_i\}$. We define the *t-extended density operator* to be the following.

$$S_t = \sum_k \pi_k \left|\psi_k\right\rangle^{\otimes t} \left\langle\psi_k\right|^{\otimes t} \tag{13}$$

From Theorem 1, using the equality condition, one can see that S_t with $\{\Pi_k, \pi_k\}_{k=1}^n$, $n \geq {}_{t+d-1}C_{d-1}$ is a generalized t-design if and only if

$$\mathrm{Tr}(S_t^2) = \frac{t!(d-1)!}{(t+d-1)!}, \tag{14}$$

and this is a global minimum of $\mathrm{Tr}(S_t^2)$. The sequence of derivation we have shown so far resembles that of a *spherical t-design* in [3]. One major difference is that in [3], the probabilities corresponding to the states were uniform while here the condition is relaxed.

4 Generalized 2-Design and Ensembles Achieving Minimal Quantumness

Let us start with the definition of average fidelity.

$$F_{\mathcal{P}}(\mathcal{E},\mathcal{M}) = \sum_{b,i} \pi_i \mathrm{Tr}(\Pi_i E_b)\mathrm{Tr}(\Pi_i \sigma_b) \tag{15}$$

Let $\tilde{\rho}$ be the 2-extended density operator $\sum_i \pi_i \Pi_i^{\otimes 2}$ and let $X \equiv \sum_b E_b \otimes \sigma_b$. We can interprete this as E_b living on a *measurement operator space* and σ_b living on a *reconstruction operator space*. One can see that the average fidelity can be simply rewritten in the following expression.

$$F_{\mathcal{P}}(\mathcal{E},\mathcal{M}) = \mathrm{Tr}(\tilde{\rho}X) \tag{16}$$

POVM elements E_b can be expressed as a projection of subnormalized vectors, and the state-reconstruction density operator σ_b can be expressed as a linear sum of normalized projection operators $\sum_i p_i \left|\phi_i\right\rangle\left\langle\phi_i\right|$. Consider an ensemble which forms a generalized 2-design. We use the state reconstruction strategy $\{E_b = \lambda_b \left|\phi_b\right\rangle\left\langle\phi_b\right| \mid \sum_b E_b = I, 0 < \lambda_b \leq 1\}$, $\{\sigma_b = \sum_k p_k \left|\varphi_k^b\right\rangle\left\langle\varphi_k^b\right| \mid \sum_k p_k = 1\}$, which captures any possibility. Therefore, the corresponding average fidelity becomes the following.

$$\int d\psi \sum_{b,k} p_k \lambda_b \left|\langle\phi_b|\psi\rangle\right|^2 \left|\langle\varphi_k^b|\psi\rangle\right|^2 \tag{17}$$

Lemma 1 *Ensembles which form generalized* 2*-design achieves minimal quantumness.*

Proof Notice the following Cauchy-Schwarz inequality for nonnegative functions f and g over a measure Ω.

$$\int d\Omega f(\Omega)g(\Omega) \leq \sqrt{\int d\Omega f(\Omega)^2 \int d\Omega g(\Omega)^2} \tag{18}$$

We can directly apply this inequality to Eq.17. Since the equality holds if and only if $\left|\langle\phi_b|\psi\rangle\right|^2 = \left|\langle\varphi_k^b|\psi\rangle\right|^2$ for all $\left|\psi\right\rangle$, $\left|\phi_b\right\rangle$ and $\left|\varphi_k^b\right\rangle$ must be same up to a phase for all b and k. When this equality holds, we obtain an expression for the average fidelity,

$$F_{\mathcal{P}}(\mathcal{E},\mathcal{M}) = d\int d\psi \left|\langle\psi_i|\psi\rangle\right|^4 \tag{19}$$

$$= \tfrac{2}{d+1} \tag{20}$$

where $\left|\psi_i\right\rangle$ is a dummy state for integration. Therefore, any ensemble which is a generalized 2-design achieves minimal quantumness. Furthermore, the optimal strategy is to measure the states with rank-1 POVM and reconstruct the corresponding states. This agrees with the result from [4].

Assume the converse of this statement is incorrect. This means the 2*-extended density operator* generated by the ensemble is not identical to the projector onto the symmetric subspace. This quantity can be represented as the following equation.

$$\Delta\tilde{\rho} = \tilde{\rho} - \frac{2}{d(d+1)}\Pi_{sym} \tag{21}$$

Note that by the construction of $\Delta\tilde{\rho}$, it is traceless. Similarly, we may reexpress $\tilde{X}(\tilde{\rho})$ as $\frac{2}{d+1}\Pi_{sym} + \tilde{X}'(\tilde{\rho})$. In this case, we have a stronger relation then $\mathrm{Tr}(\tilde{X}') = 0$. Since

$\mathrm{Tr}_2(X) \equiv \sum_b E_b \mathrm{Tr}(\sigma_b) = I$ for any X, we get $\mathrm{Tr}_2(\tilde{X}') = 0$. Now we can express the accessible fidelity in a following form, owing to the fact that $\Delta\tilde{\rho}$ and $\tilde{X}'$ are traceless.

$$F_{\mathcal{P}} = \frac{2}{d+1} + \mathrm{Tr}(\Delta\tilde{\rho}\tilde{X}'(\tilde{\rho})) \tag{22}$$

Lemma 2 *If $\Delta\tilde{\rho} \neq 0$, there exists $\tilde{X}'$ such that $Tr(\Delta\tilde{\rho}\tilde{X}'(\tilde{\rho})) \neq 0$.*

Proof Since $\Delta\tilde{\rho}$ is hermitian, we can diagonalize it respect to some orthogonal basis. Since the trace is invariant under unitary transform of the operators, we work in this basis for $\tilde{X}'$ as well. Since we are working in the symmetric subspace, we denote each of the diagonal entries as (i,j) where $(i,j) = (j,i)$. Suppose each diagonal entries of $\Delta\tilde{\rho}$ are denoted as $\alpha_{(i,j)}$. The following relation must hold, since $\mathrm{Tr}(\Delta\tilde{\rho}) = 0$.

$$\sum_{i=1}^{d}\sum_{j=1}^{i} \alpha_{(i,j)} = 0 \tag{23}$$

Let us denote the diagonal entries of $\tilde{X}'$ as $\beta_{(i,j)}$. Due to the constraint $\mathrm{Tr}_2(\tilde{X}') = 0$, we obtain a system of linear constraints.

$$\beta_{(i,i)} + \frac{1}{2}\sum_{j=1, j\neq i}^{d} \beta_{(i,j)} = 0 \quad \forall i \tag{24}$$

Using this equation, we can deduce the way to set $\mathrm{Tr}(\Delta\tilde{\rho}\tilde{X}'(\tilde{\rho}))$ to 0.

$$\mathrm{Tr}(\Delta\tilde{\rho}\tilde{X}'(\tilde{\rho})) = \textstyle\sum_{(i,j)} \alpha_{(i,j)}\beta_{(i,j)} \tag{25}$$

$$= \textstyle\sum_{i=1}^{d}\sum_{j=1}^{i-1} \alpha_{(i,j)}\beta_{(i,j)} + \sum_{i=1}^{d} \alpha_{(i,i)}\beta_{(i,i)} \tag{26}$$

$$= \tfrac{1}{2}\textstyle\sum_{i=1}^{d}\sum_{j=1, j\neq i}^{d} (\alpha_{(i,j)} - \alpha_{(i,i)})\beta_{(i,j)} = 0 \tag{27}$$

Note that $\sum_{(i,j)} = \sum_{i=1}^{d}\sum_{j=1}^{i}$. Since $\beta_{(i,j)}$ can have nontrivial values, to set $\mathrm{Tr}(\Delta\tilde{\rho}\tilde{X}'(\tilde{\rho}))$ to zero regardless of the choice of $\beta_{(i,j)}$, $\alpha_{(i,j)}$ must be a constant respect to the indices i and j. Since $\sum \alpha_{(i,j)} = 0$ we can conclude that $\alpha_{(i,j)} = 0$ for all i and j. Therefore, unless $\Delta\tilde{\rho}$ is not equal to the null operator, we can always find an operator which satisfies $\mathrm{Tr}(\Delta\tilde{\rho}\tilde{X}'(\tilde{\rho})) \neq 0$. This ends the proof.

From Lemma 1 and Lemma 2, we can conclude the following.

Theorem 2 *Ensemble $\mathcal{P}(\Pi, \pi)$ is a generalized 2-design if and only if it achieves the minimal quantumness.*

Proof In Lemma 1, we already proved that ensembles which are generalized 2-design achieve minimal quantumness. Therefore, the remaining part of the proof is the converse of that statement. Note that we can prevent X' from violating the positivity of the operator $\tilde{X}$, by dividing $\tilde{X}'$ with a sufficiently large number. If it gives negative result, we can simply change its sign, keeping the entries small enough to prevent the violation of positivity. This means that if the ensemble does not form a generalized 2-design, we can always *do better* than the cases corresponding to the minimal quantumness. Therefore, all ensembles which achieve minimal quantumness are necessarily generalized 2-design.

Furthermore, minimal number of states needed for generalized 2-design is d^2, and the logic follows very closely that of [3]. In [3], the authors show that the minimal number of states needed for spherical 2-design is d^2 by proving the following relation.

$$\int_{\mathcal{S}_d} d\psi \left|\psi\right\rangle \left\langle\psi\right| A \left|\psi\right\rangle \left\langle\psi\right| = \frac{1}{d(d+1)}(A + \mathrm{Tr}(A)I) \quad \forall A \in \mathcal{B}(\mathcal{H}_d), \tag{28}$$

where $\mathcal{B}(\mathcal{H}_d)$ is a set of linear operators on $\mathcal{H}_d$.

Note that the left hand side of Eq.28 is identical to the action of generalized 2-design on an element of $\mathcal{B}(\mathcal{H}_d)$. Since the right hand side is clearly a rank-d^2 operator, it follows that the minimal number of states needed is d^2. For a detailed proof, we recommend to consult section 2 of [3].

5 Generalized 2-design with d^2 elements are Spherical 2-design

Although spherical 2-design is a proper subset of generalized 2-design, these two sets coincide if we only take care of the minimal sets. Since generalized 2-design is automatically a generalized 1-design as well, by using Lemma 1, we may find following two formulae.

$$\mathrm{Tr}(\tilde{\rho}^2) = \tfrac{2}{d(d+1)} \tag{29}$$

$$\mathrm{Tr}(\rho^2) = \tfrac{1}{d} \tag{30}$$

Setting $\lambda_{jk} \equiv \left|\langle\psi_j|\psi_k\rangle\right|^2$ for $j \neq k$, we can reformulate Eq.29 and Eq.30.

$$\textstyle\sum_{j\neq k} p_j p_k \lambda_{jk}^2 + \sum_k p_k^2 = \frac{2}{d(d+1)} \tag{31}$$

$$\textstyle\sum_{j\neq k} p_j p_k \lambda_{jk} + \sum_k p_k^2 = \frac{1}{d} \tag{32}$$

Now let us find the range that the quadratic form $\sum_{j\neq k} p_j p_k \lambda_{jk}^2$ can have under the constraint of Eq.32. By using Lagrange's multiplier method, the minimum can be found at points $\lambda_{jk} = \frac{1}{1-\alpha}(\frac{1}{d} - \alpha)$, where $\alpha \equiv \sum_k p_k^2$. If the number of ensembles is d^2, a bound on α becomes $\frac{1}{d^2} \leq \alpha \leq 1$, where the former bound is achieved if and only if $p_j = \frac{1}{d^2}$ for all j. From this inequality, we can derive the following result.

$$\sum_{j\neq k} p_j p_k \lambda_{jk}^2 + \sum_k p_k^2 \geq \frac{2}{d(d+1)} \tag{33}$$

The bound is achieved if and only if $\alpha = \frac{1}{d^2}$, $\lambda_{jk} = \frac{1}{d+1}$. Therefore, if a generalized 2-design exists for d^2 elements, it must be a spherical 2-design as well. Note that this argument cannot be applied to the number of ensembles not equal to d^2.

6 Discussion

In this paper, we have shown that ensembles achieveing minimal quantumness are generalized 2-design and vice versa. As a corollary, we also showed that the minimal number of states needed to achieve minimal quantumness is d^2. Furthermore, for these ensembles, their optimal strategy is to measure states and simply reconstruct the corresponding states, as shown in [4]. However, all the ensembles that have been known so far to achieve the minimal quantumness are spherical 2-design. These are the unitarily invariant ensemble, SIC-ensemble,

and ensemble formed by a uniform distribution of mutually unbiased basis(MUB). [1][3][4][8] Although it is possible to construct a generalized 2-design which is not a spherical 2-design by merging two different spherical 2-designs, it will be interesting to see ensembles which achieve a minimal quantumness and cannot be decomposed in such a trivial way. Whether such examples exist or not is left as an open problem.

Also, by showing the equivalence between the generalized 2-design and spherical 2-design for d^2 ensembles, we found an equivalent definition of the SIC-POVM in a totally different point of view. This means we have a new way to prove the existence of the SIC-POVM in all dimensions that has been unseen among the previous literatures. The existence of SIC-POVM will follow if the minimal quantumness can be achieved with only d^2 vectors. The usefulness of this approach has not been investigated much at this point, but we are hoping to see positive results.

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References

1. C. A. Fuchs and M. Sasaki, Squeezing Quantum Information through a Classical Channel: Measuring the "Quantumness" of a Set of Quantum States, Quant. Info. Comp. **3**, 377–404 (2003); see also quant-ph/0302092.
2. A. J. Scott, Tight Informationally Complete Quantum Measurements, J. Phys. A **39**, 13507 (2006); see also quant-ph/0604049.
3. J. M. Renes, R. Blume-Kohout, A. J. Scott, and C. M. Caves, Symmetric Informationally Complete Quantum Measurements, J. Math. Phys. **45**, 2171 (2004); see also quant-ph/0310075.
4. C. A. Fuchs (2004), On the Quantumness of the Hilbert Space, Quant. Info. Comp. **4**, 467–478 (2004); see also quant-ph/0404122.
5. G. Zauner, Quantendesigns, Grundzuge einer nichtkommutativen Designtheorie, Doctorial thesis, University of Vienna, (1999).
6. D. M. Appleby, SIC-POVMs and the Extended Clifford Group, J. Math. Phys. **46**, 052107 (2005); see also quant-ph/0412001.
7. J. J. Benedetto and M. Fickus, Finite Normalized Tight Frames. Advances in Computational Mathematics, 18, 357–385 (2003).
8. A. Klappenecker and M. Rötteler, Mutually Unbiased Bases are Complex Projective 2-Designs, Proc. 2005 IEEE International Symposium on Information Theory, Adelaide, Australia, 1740–1744 (2005); see also quant-ph/0502031.